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Matroid Basis Graphs. II*

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Several graph-theoretic notions applied to matroid basis graphs in the preceding paper are now tied more specifically to aspects of matroids themselves. Factorizations of basis graphs and disconnections of neighborhood subgraphs are related to matroid separations. Matroids are characterized whose basis graphs have only one or two of the three types of common neighbor subgraphs. The notion of leveling is generalized and related to matroid sums, minors, and duals. Also, the problem of characterizing regular and graphic matroids through their basis graphs is discussed. Throughout, many results are obtained quite easily with the aid of certain pseudo-combivalence systems of 0-1 matrices.

1. Introduction and Preliminaries

In [7] we characterized matroid basis graphs. Three concepts which played important roles in our main characterization were neighborhood subgraphs, common neighbor subgraphs, and levelings. We will relate here these features of basis graphs to features of the matroids they represent. In Section 3 we show that a matroid is separable iff some neighborhood subgraph of its basis graph is disconnected, and also iff the whole basis graph is a direct product. Similar results have been obtained by others [1, 4] but not, we think, so concisely. In Section 4 we analyze matroids whose basis graphs do not contain all three types of common neighbor subgraphs. The most interesting of these results is that a matroid is binary iff its basis graph contains no octahedra. In Section 5 the notion of leveling is generalized and the special structure of the top and bottom levels (the polars) is explored. Finally, in Section 6 we ask whether there

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are basis graph characterizations for such classes as regular and graphic matroids. We obtain some answers, but question their usefulness.

To each matroid one may associate a set of 0-1 matrices closely related to the cycle and cocycle matrices of graph theory. These matrices were first studied systematically by Yoseloff [15]. As systems they have almost as much structure as the combivalent matrices introduced by Tucker [11], whence we call them pseudo-combivalent. We introduce these matrix notions in Section 2 and use them continually thereafter. Although their use is by no means necessary, we have found that they greatly simplify the proofs and sometimes the statements of our theorems.

We use the Roman numeral I to refer to [7]. For instance, Theorem I.2.2 is Theorem 2.2 of that paper. We make the present paper reasonably self-contained by including several items from I below, sometimes slightly reworded.

A *matroid* $\mathcal{M}(E, \mathcal{B})$ is a finite set of *elements* E and a collection of *bases* $\mathcal{B}$, all subsets of E, which satisfy the following

EXCHANGE AXIOM. *For all $B, B' \in \mathcal{B}$ and $e' \in B' - B$, there exists $e \in B - B'$ such that $B - e + e' \in \mathcal{B}$.*

All $B \in \mathcal{B}$ necessarily have the same cardinality, called the *rank*. $\mathcal{M}$ is *full* if $\mathcal{B}$ consists of all subsets of E with a given rank. A matroid $\mathcal{M}'(E', \mathcal{B}')$ is a *submatroid* of $\mathcal{M}(E, \mathcal{B})$ if $E' = E$ and $\mathcal{B}' \subset \mathcal{B}$. $\mathcal{M}(E, \mathcal{B})$ and $\mathcal{M}'(E', \mathcal{B}')$ are *isomorphic* ($\mathcal{M} \approx \mathcal{M}'$) if there is a bijection $f: E \to E'$ such that $B \in \mathcal{B}$ iff $f(B) \in \mathcal{B}'$.

$G(\mathcal{V}, \mathcal{E})$ shall denote a finite graph with vertices $\mathcal{V} = \mathcal{V}(G)$ and edges $\mathcal{E} = \mathcal{E}(G)$. Neither loops nor multiple edges are allowed. For $v \in \mathcal{V}$, the *neighborhood subgraph* $N(v)$ is the induced subgraph on all vertices adjacent to v. If the shortest path from v to v' has length 2, i.e., $\delta(v, v') = 2$, then the induced subgraph on v, v' and all vertices adjacent to both is called the *common neighbor subgraph* $\mathrm{CN}(v, v')$, or simply a CN. A *leveling* of G from v_0 is a partition of $\mathcal{V}$ into

$$\mathcal{V}_k = \{v \mid \delta(v, v_0) = k\}, \quad k = 0, 1, \ldots .$$

As usual, G and G' are isomorphic ($G \approx G'$) if there is a bijection

$$\mathcal{V}(G) \to \mathcal{V}(G')$$

which preserves adjacency.

A graph is *properly labeled* if each vertex is labeled with a finite set (in which case we write B, B', $\mathcal{B}$ instead of v, v', $\mathcal{V}$) and furthermore, B, B' are adjacent iff $| B - B' | = | B' - B | = 1$. G is the *labeled basis graph*

$BG(\mathcal{M})$, also called $BG(E, \mathcal{B})$, if it is properly labeled and its labels are the bases of the matroid $\mathcal{M}(E, \mathcal{B})$. G is simply a *basis graph* if it can be labeled to become some $BG(\mathcal{M})$. In any basis graph each $N(v)$ is the line graph of a bipartite graph (Lemma I.1.8) and each CN is a *square*, a *pyramid* (with square base), or an *octahedron* (Lemma I.1.4 and Fig. I.1).

THEOREM 1.1 (I.2.2). *Suppose G is connected and properly labeled. Then G is a labeled basis graph iff each* CN *is a square, pyramid or octahedron.*

Lemma I.2.4 says that, if all but one vertex of a basis graph CN is properly labeled, then there is a unique label for the remaining vertex which properly labels the whole. Using this repeatedly while working out from v one may show

LEMMA 1.2. *Suppose G is a basis graph and the induced subgraph on some v and all its neighbors is properly labeled. Then there is at most one extension of this labeling which makes G a labeled basis graph.*

In fact, such an extension always does exist. This follows from the proof of our Main Theorem I.2.1. A more direct proof has been obtained independently by Holzmann, Norton, and Tobey [4], who also give an explicit proof of Lemma 1.2.

THEOREM 1.3 (I.4.2). *Suppose $\mathcal{M}(E, \mathcal{B})$ is full. If $\mathcal{B}' \subset \mathcal{B}$ has the property that $\delta(B', B'') \geqslant 2$ for any distinct $B', B'' \in \mathcal{B}'$, then $(E, \mathcal{B} - \mathcal{B}')$ is a matroid.*

2. COMBIVALENCE AND PSEUDO-COMBIVALENCE

Let V be a finite set of vectors (not necessarily distinct) from some vector space. Let $\mathcal{X}$ be the collection of all maximal independent sets $X \subset V$. Then it is easily verified that $(V, \mathcal{X})$ is a matroid. Indeed, in many older linear algebra texts the exchange axiom in this situation is explicitly singled out as the Steinitz Exchange Principle.

Any matroid which is isomorphic to some such vector matroid is said to be *representable*. The important problem of characterizing representable matroids is still unsolved. Ingleton [5] gives a good survey of current knowledge.

A vector matroid $\mathcal{M}(V, \mathcal{X})$ is usually represented as a matrix by picking some basis for the underlying space and writing each $v \in V$ as a column vector over this basis. However, we will represent $\mathcal{M}$ by a whole system of smaller matrices. For $X = \{x_1, ..., x_m\}$ in $\mathscr{X}$, let $Y = \{y_1, ..., y_n\} = V - X$. Then there are unique constants a_{ij} such that

$$y_j = \sum_{i=1}^{m} a_{ij} x_i , \qquad j = 1, 2, ..., n. \tag{1}$$

Schematically we write

$$\begin{array}{c|ccc|}
\cline{2-4}
x_1 & & \vdots & \\
\vdots & \cdots & a_{ij} & \cdots \\
x_m & & \vdots & \\
\cline{2-4}
\multicolumn{1}{c}{} & \multicolumn{3}{c}{= y_1 \;\cdots\; = y_n}
\end{array} . \tag{2}$$

We refer to (2), border symbols included, as the *reduced matrix* $M(X)$ of $\mathscr{M}(V, \mathscr{X})$. For each X, $M(X)$ is clearly unique up to order of rows and order of columns.

For each $\mathscr{M}(V, \mathscr{X})$ the set $\{M(X) | X \in \mathscr{X}\}$ is called a *combivalence system*, and the matrices therein are *combivalent*. Combivalence was introduced, with an equivalent definition, by A. W. Tucker [11]. He has applied the concept to linear programming, game theory, and graph theory [8, 12].

With X, Y, a_{ij} as before, we have that $X' = X - x_k + y_l$ is in $\mathscr{X}$ iff $a_{kl} \neq 0$. If $X' \in \mathscr{X}$, and $M(X') = [b_{ij}]$, then

$$\begin{aligned}
b_{kl} &= 1/a_{kl} ; \\
b_{il} &= -a_{il}/a_{kl} , \qquad i \neq k; \\
b_{kj} &= a_{kj}/a_{kl} , \qquad j \neq l; \\
b_{ij} &= a_{ij} - (a_{kj}a_{il}/a_{kl}), \qquad i \neq k, \qquad j \neq l.
\end{aligned} \tag{3}$$

The ordering of $M(X')$ used here is the one obtained from (2) by simply interchanging x_k and y_l. The proof of (3) is by elementary algebra. The form of (3) is easily remembered by the schema

$$\boxed{\begin{matrix} p^* & q \\ r & s \end{matrix}} \longrightarrow \boxed{\begin{matrix} 1/p & q/p \\ -r/p & s - (rq/p) \end{matrix}} . \tag{4}$$

The (*) marks the entry indexed by the vectors to be exchanged. We will refer to the entries on the right as p', q', r', s'.

Anyone familiar with linear programming will recognize (3) and (4) as embodying the standard pivoting rules. As usual, the operation denoted in (4) will be called a *pivot step* with *pivot* p, and a series of such steps will be a *pivot sequence*. Indeed, in I we applied pivot terminology to the general matroid situation, and when we introduce pseudo-combivalence below we will apply it there too.

$\mathcal{M}$ is said to be *binary* if it is representable over the field $F_2 = \{0, 1\}$. For binary matroids (4) takes a particularly simple form. We have $p' = p = 1$, $q' = q$, $r' = r$ in all cases, and $s' = s$ except for

$$\begin{array}{|cc|}\hline 1^* & 1 \\ 1 & 0 \\ \hline\end{array} \longrightarrow \begin{array}{|cc|}\hline 1 & 1 \\ 1 & 1 \\ \hline\end{array}, \tag{5}$$

$$\begin{array}{|cc|}\hline 1^* & 1 \\ 1 & 1 \\ \hline\end{array} \longrightarrow \begin{array}{|cc|}\hline 1 & 1 \\ 1 & 0 \\ \hline\end{array}. \tag{6}$$

A matroid is *graphic* if its bases are the edge sets of the spanning forests of some graph. It is well known that every graphic matroid is binary. If $\mathcal{M}(V, \mathscr{X})$ is graphic and $M(X) = [a_{ij}]$, then

$$\{y_l\} \cup \{x_k \mid a_{kl} = 1\}$$

is just the fundamental cycle for forest X and chord y_l. Also

$$\{x_k\} \cup \{y_l \mid a_{kl} = 1\}$$

is just the fundamental cocycle (cut-set) for the forest X and twig x_k. Thus $M(X)$ is closely related to the usual cycle and cocycle matrices. Indeed, it is a submatrix of either and determines both [3].

Given any $\mathcal{M}(E, \mathscr{B})$, not necessarily binary, we may still attach a 0-1 matrix to each B as follows: create a row for each $b \in B$ and a column for each $c \in E - B$, and let the (b, c) entry be 1 iff $B - b + c \in \mathscr{B}$. Clearly this matrix is just $M(B)$ if $\mathcal{M}$ is binary. In all cases we call it the reduced *circuit matrix* $C(B)$.

Surprisingly, the set of circuit matrices of a non-binary matroid behaves, with just one exception, like a combivalence system. This result, first obtained by Yoseloff [15], will now be slightly reformulated and given a simple proof using basis graphs.

DEFINITION 2.1. *A pseudo-combivalence system* is a collection $\mathscr{P}$ of

0-1 matrices, each with its rows and columns indexed by a fixed set E, such that

(1) For each $B \subset E$ there exists (up to order) at most one $P(B) \in \mathscr{P}$ with rows indexed by B,

(2) Suppose $P(B)$ exists and $B' = B - b + c$, where $b \in B$ and $c \in E - B$. Then

- (i) $P(B')$ exists iff the (b, c) entry of $P(B)$ is 1;
- (ii) if $P(B')$ exists, its entries are determined from those of $P(B)$ by schema (4), except that both (6) and

$$\begin{bmatrix} 1^* & 1 \\ 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{7}$$

are allowed; and

(3) Any $P(B'')$ can be reached from any $P(B)$ by a pivot sequence.

THEOREM 2.2. *A set of matrices is a pseudo-combivalence system if and only if it is the set of circuit matrices of some matroid.*

Proof. Sufficiency: Given $\mathscr{M}(E, \mathscr{B})$, let $P(B) = C(B)$. Conditions (1) and (2i) are immediate from the definition of circuit matrix. As for (2ii), let

$$\begin{array}{c|cc|} b & p & q \\ b' & r & s \\ \hline & c & c' \end{array}$$

be any 2×2 submatrix of $C(B)$. It corresponds to the CN of Figure 1(a) in the basis graph of the full matroid on E of which $\mathscr{M}$ is a submatroid. Vertex b/c, that is, $B - b + c$, is actually in $BG(\mathscr{M})$ iff $p = 1$, and so forth. Should $p = 1$, we may pivot on p to obtain

$$\begin{array}{c|cc|} c & p' & q' \\ b' & r' & s' \\ \hline & b & c' \end{array}.$$

This corresponds to the same CN as before, but now the matrix entries are attached as in Figure 1(b). Since by assumption both B and b/c exist,

$p' = p = 1$. Since q, q' refer to the same vertex, $q' = q$. Likewise $r' = r$. We now need only determine when s' can or must differ from s. If $s = 1$, consider CN($b/c, b'/c'$) in $BG(\mathcal{M})$. It must be a square, pyramid, or octahedron. Unless all the middle level vertices exist, we must have $s' = 1$. When they do all exist, we can, but need not, have $s' = 0$. This gives (6) and (7). Now suppose $s = 0$. If $q = r = 1$, by considering

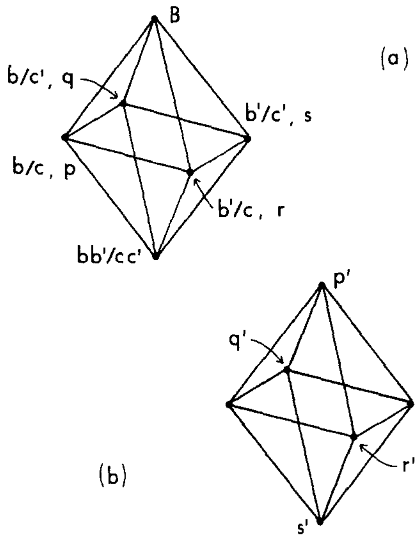


FIG. 1. One CN as related to two submatrices.

$CN(b/c', b'/c)$ we see that s' must be 1. This gives (5). In the remaining cases s' must be 0: otherwise $CN(B, bb'/cc')$ is improper. Thus (2ii) is proved. Finally (3) follows, as usual, from the exchange axiom.

Necessity: Consider the properly labeled graph on $\mathcal{B} = \{B \mid P(B) \in \mathcal{P}\}$. It is connected by condition (3). Suppose $\delta(B, B'') = 2$. There must be at least one intermediate vertex B'. We may assume $B' = b/c$, $B'' = bb'/cc'$. Then, by the same analysis as above for the case $s' = 1$, we get that $CN(B, B'')$ is a square, pyramid, or octahedron. By Theorem 1.1, $(E, \mathcal{B})$ is a matroid. By condition (2i), $P(B) = C(B)$. ∎

Remark. Condition (2ii) does not say that one can choose between (6) and (7) at will. For instance, if one insisted on choosing (7) always, one would have to allow

$$\begin{array}{c|cc|} b & 1^* & 1 \\ b' & 1 & 0 \\ \hline & c & c' \end{array} \longrightarrow \begin{array}{c|cc|} c & 1^* & 1 \\ b' & 1 & 1 \\ \hline & b & c' \end{array} \longrightarrow \begin{array}{c|cc|} b & 1 & 1 \\ b' & 1 & 1 \\ \hline & c & c' \end{array};$$

but this violates condition (1). It would be interesting to find some rules for choosing so that one could

(A) pick any 0-1 matrix,

(B) pivot in it until no new matrices occur, using the same one rule whenever a choice arises,

and thereby attain a pseudo-combivalence system. Clearly, condition (1) is the only condition that might be violated. The rule "always use (7)" does not work. The rule "always use (6)" does, for it strengthens pseudo-combivalence to combivalence. Unfortunately, this is the only "good" rule we know.

3. Sums, Parts, Products and Duals

Given $\mathscr{M}_1(E_1, \mathscr{B}_1)$ and $\mathscr{M}_2(E_2, \mathscr{B}_2)$, where $E_1 \cap E_2 = \varnothing$, the *sum* $\mathscr{M} = \mathscr{M}_1 + \mathscr{M}_2$ is the matroid with elements $E_1 \cup E_2$ and bases

$$\mathscr{B}_1 + \mathscr{B}_2 = \{B_1 \cup B_2 \mid B_1 \in \mathscr{B}_1, B_2 \in \mathscr{B}_2\}.$$

$\mathscr{M}(E, \mathscr{B})$ is *trivial* if $E = \varnothing$ and $B = \{\varnothing\}$. If $\mathscr{M} = \mathscr{M}_1 + \mathscr{M}_2$ and neither $\mathscr{M}_1$ nor $\mathscr{M}_2$ is trivial, we say that $\mathscr{M}$ is *separable* with *components* $\mathscr{M}_1$, $\mathscr{M}_2$.

Given $\mathscr{M}(E, \mathscr{B})$, $e \in E$ is a *loop* if it is outside every basis. It is a *coloop* if it is in every basis. Let L and C be the loop and coloop sets of $\mathscr{M}$. Let

$$E_S = E - (L \cup C), \qquad \mathscr{B}_S = \{B \cap E_S \mid B \in \mathscr{B}\}.$$

Then

$$\begin{aligned} \mathscr{M}_I &= (L \cup C, \{C\}), \\ \mathscr{M}_S &= (E_S, \mathscr{B}_S), \end{aligned} \tag{8}$$

are matroids and we call them the *insignificant* and *significant* parts of $\mathscr{M}$. The names are justified by

LEMMA 3.1. *$\mathscr{M} = \mathscr{M}_I + \mathscr{M}_S$, $BG(\mathscr{M}_I)$ is a single vertex, and*

$$BG(\mathscr{M}_S) \approx BG(\mathscr{M}).$$

Proof. The first claim follows because $B = (B - C) \cup C$ for all $B \in \mathscr{B}$. The second is true by definition. The third follows from the basis bijection $B - C \to B$. ∎

Given $G_1(\mathscr{V}_1, \mathscr{E}_1)$ and $G_2(\mathscr{V}_2, \mathscr{E}_2)$, the *product* $G = G_1 \times G_2$ is the graph with vertices $\mathscr{V}_1 \times \mathscr{V}_2$ and edges

$$\{(u, v)(u', v) \mid uu' \in \mathscr{E}_1\} \cup \{(u, v)(u, v') \mid vv' \in \mathscr{E}_2\}.$$

$G(\mathscr{V}, \mathscr{E})$ is *trivial* if it consists of a single vertex. If $G \approx G_1 \times G_2$ and neither G_1 nor G_2 is trivial, we say G is *composite* with *factors* G_1, G_2.

Finally, a bordered matrix M is a *sum of blocks* M_1, M_2 if up to order

$$M = \begin{array}{c|c:c|} \multicolumn{1}{c}{} & \multicolumn{2}{c}{} \\ \cline{2-3} B_1 & M_1 & 0 \\ \hdashline B_2 & 0 & M_2 \\ \cline{2-3} \multicolumn{1}{c}{} & \multicolumn{1}{c}{C_1} & \multicolumn{1}{c}{C_2} \end{array}, \tag{9}$$

and neither M_1 nor M_2 is empty or all zeros. B_1 represents a set of row indices, etc., and each 0 represents a submatrix filled with zeros. If M is the sum of several blocks, we write $M = \sum M_i$.

THEOREM 3.2. *Suppose $\mathscr{M}(E, \mathscr{B})$ has neither loops nor coloops. Then the following are equivalent*:

(1) *$\mathscr{M}$ is separable*;

(2) *$BG(\mathscr{M})$ is composite*;

(3) *for some $B \in \mathscr{B}$, the subgraph $N(B)$ of $BG(\mathscr{M})$ is disconnected*;

(4) *for some B, $C(B)$ is the sum of blocks.*

Proof. (1) $\Rightarrow$ (2). Suppose $\mathscr{M} = \mathscr{M}_1(E_1, \mathscr{B}_1) + \mathscr{M}_2(E_2, \mathscr{B}_2)$. Then, $BG(\mathscr{M}) \approx BG(\mathscr{M}_1) \times BG(\mathscr{M}_2)$ by the basis bijection $B_1 \cup B_2 \to (B_1, B_2)$. Moreover, any loop or coloop of $\mathscr{M}_i$, $i = 1$ or 2, would also be one in $\mathscr{M}$; hence $BG(\mathscr{M}_i)$ is not trivial.

(2) $\Rightarrow$ (3). Suppose $BG(\mathscr{M}) \approx G_1 \times G_2$ where neither G_i is trivial. Take *any* $B \in \mathscr{B}$. It corresponds to some (v_1, v_2) in $G_1 \times G_2$. By the definition of graph product, the vertex sets in $BG(\mathscr{M})$ corresponding to

$$\{(v_1, v) \mid vv_2 \in \mathscr{E}(G_2)\}, \qquad \{(v, v_2) \mid vv_1 \in \mathscr{E}(G_1)\} \tag{10}$$

are both non-empty and disconnect $N(B)$.

(3) $\Rightarrow$ (4). By definition of circuit matrix, each vertex in $N(B)$ corresponds to a 1 in $C(B)$; moreover, two vertices of $N(B)$ are adjacent iff their 1's are in the same row or column. Also, since $\mathscr{M}$ has no loops (coloops), $C(B)$ has no zero columns (rows). Now if vertex sets $\mathscr{B}_1, \mathscr{B}_2$

disconnect $N(B)$, let B_i, C_i be the rows and columns of $C(B)$ in which 1's corresponding to $\mathscr{B}_i$ occur. By the above, these partition B and $C = E - B$ and we have (9), where neither M_i is empty or a zero matrix.

(4) $\Rightarrow$ (1). Suppose some $C(B_0)$ is as in (9). By the pseudo-combivalence rules, no pivot in the upper left (lower right) affects any of the other three quadrants. In other words, any basis can be obtained from B_0 by making exchanges in $E_1 = B_1 \cup C_1$ and $E_2 = B_2 \cup C_2$ independently. Let

$$\mathscr{B}_i = \{B \cap E_i \mid B \in \mathscr{B}\}.$$

Then $(E_1, \mathscr{B}_1)$ and $(E_2, \mathscr{B}_2)$ both satisfy the exchange axiom and are thus matroids $\mathscr{M}_1, \mathscr{M}_2$. Since $\mathscr{B} = \mathscr{B}_1 + \mathscr{B}_2$, $\mathscr{M} = \mathscr{M}_1 + \mathscr{M}_2$. ∎

If $\mathscr{M}$ is allowed to have loops and coloops, a nontrivial separation of $\mathscr{M}$ still corresponds to a factorization of $BG(\mathscr{M})$ and a breakdown of each $C(B)$ into blocks, but the factorization may be trivial and one of the blocks may be a zero matrix or empty (specifically, of size $k \times 0$ or $0 \times l$). Lemma 3.1 provides an example.

We can make the relationship between graph products and matroid sums more precise. Suppose $G \approx G_1 + G_2$ by a vertex bijection f. For any $v_2 \in \mathscr{V}(G_2)$, G_1 is isomorphic to the induced subgraph of G on $\{f(v, v_2) | \, v \in \mathscr{V}(G_1)\}$. We call this a *natural image* of G_1 by f. Likewise there are natural images of G_2.

THEOREM 3.3. *If $BG(\mathscr{M}) \approx G_1 \times G_2$ by f, then for any natural images G_i' of G_i by f, $i = 1, 2$,*

(1) *the vertices of G_i' are labeled with the bases of a matroid $\mathscr{M}_i$, and,*

(2) *except for loops and coloops of the $\mathscr{M}_i$, $\mathscr{M} = \mathscr{M}_1 + \mathscr{M}_2$.*

Proof. Without loss of generality, we may assume that $\mathscr{M}$ is without loops or coloops and that neither G_i is trivial. Now, since G_i' is an induced subgraph of a properly labeled graph, it is properly labeled. Since it is a factor of a connected graph, it is connected. Also because it is a factor, if $\delta(v, v') = 2$ in G_i', then all paths between v, v' in $BG(\mathscr{M})$ and at least partly outside of G_i' have length greater than 2 (in fact, at least 4). Thus $\mathrm{CN}(v, v')$ in G_i' is the same as $\mathrm{CN}(v, v')$ in $BG(\mathscr{M})$. By Theorem 1.1, condition (1) obtains. If $\mathscr{M} = (E, \mathscr{B})$, we may write $\mathscr{M}_i = (E, \mathscr{B}_i)$.

Let $B_0 = f(v_1, v_2)$ be the unique vertex common to G_1' and G_2'. Consider the intersections of the G_i' with $N(B_0)$. These $N_i(B_0)$ are just the images by f of the sets (10) and thus disconnect $N(B_0)$. By (3) $\Rightarrow$ (4) of

Theorem 3.2, there are partitions $E = E_1 \cup E_2$, $E_1 = B_1 \cup C_1$, $E_2 = B_2 \cup C_2$ such that all labels in $N_1(B_0)$ are of the form

$$B_1' \cup B_2, \quad B_1' \subset E_1, \tag{11}$$

and all labels in $N_2(B_0)$ are of the form

$$B_1 \cup B_2', \qquad B_2' \subset E_2.$$

In fact, all labels in G_1' are as in (11): for, if some vertex in G_1' had $c_2 \in C_2$ in its label, or lacked $b_2 \in B_2$, then, by the exchange axiom applied to $\mathscr{M}_1$, so would some label in $N_1(B_0)$. Thus the elements of $B_2 \cup C_2$ are coloops and loops of $\mathscr{M}_1$, and they all may be deleted to obtain a matroid $\mathscr{M}_1'(E_1, \mathscr{B}_1')$. Similarly, by deleting $B_1 \cup C_1$ from $\mathscr{M}_2$ we obtain

$$\mathscr{M}_2'(E_2, \mathscr{B}_2').$$

Finally, label $BG(\mathscr{M})$ using $\mathscr{M}_1' + \mathscr{M}_2'$. That is, if u_i in G_i has label B_i', label $f(u_1, u_2)$ with $B_1' \cup B_2'$. This labeling agrees with $\mathscr{M}$ on B_0 and $N(B_0)$. By Lemma 1.2, $\mathscr{M} = \mathscr{M}_1' + \mathscr{M}_2'$. ∎

This theorem can be obtained more directly from the previous one by using some general (but messy to prove) graph factorization uniqueness results of Sabidussi [10].

The *dual* $\mathscr{M}^*$ of $\mathscr{M}(E, \mathscr{B})$ is the matroid with elements E and bases $\mathscr{B}^* = \{E - B \mid B \in \mathscr{B}\}$. Suppose $\mathscr{M}(E, \mathscr{B})$ and $\mathscr{M}'(E', \mathscr{B}')$ have loops L, L', coloops C, C' and significant elements E_S, E_S'; see (8).Suppose further that there exist $\mathscr{M}_1(E_1, \mathscr{B}_1)$ and $\mathscr{M}_2(E_2, \mathscr{B}_2)$ where $E_1 \cup E_2 = E_S$ and

$$\mathscr{M}_S = \mathscr{M}_1 + \mathscr{M}_2. \tag{12}$$

Finally, suppose there is an element bijection $f: E_S \to E_S'$ which makes

$$\mathscr{M}_S' \approx \mathscr{M}_1 + \mathscr{M}_2^*. \tag{13}$$

Then clearly f induces an isomorphism $BG(\mathscr{M}) \approx BG(\mathscr{M}')$ by the basis bijection

$$C \cup B_1 \cup B_2 \to C' \cup f(B_1) \cup f(E_2 - B_2).$$

For completeness we state the following theorem, which has been proved by several people and published elsewhere [1, 4, 6]. We also point out a simple corollary which seems not to have been noted.

THEOREM 3.4. *Suppose $BG(\mathscr{M}) \approx BG(\mathscr{M}')$ with basis bijection g. Then there exist matroids $\mathscr{M}_1, \mathscr{M}_2$ satisfying* (12) *such that* (13) *holds with an element bijection that induces g.*

If $\mathscr{M}(E, \mathscr{B})$ is inseparable, it follows that each automorphism of $BG(\mathscr{M})$

arises from an automorphism of $\mathscr{M}$, possibly followed by a dualization. However, a dualization is not possible unless the *order* $|E|$ of $\mathscr{M}$ is exactly twice its rank. Thus, if $\Gamma(X)$ is the automorphism group of object X, we have

COROLLARY 3.5. *Suppose $\mathscr{M}$ is inseparable of order n and rank r. Then*

$$\Gamma[BG(\mathscr{M})] \approx \Gamma(\mathscr{M}),$$

except that, if $n = 2r$, it is also possible that

$$\Gamma[BG(\mathscr{M})] \approx \Gamma(\mathscr{M}) \times Z_2 . \quad \blacksquare$$

We note that, if $\mathscr{M}$ is full, $\Gamma(\mathscr{M})$ is as large as possible, namely S_n ; if, also, $n = 2r$, $\Gamma[BG(\mathscr{M})] = S_n \times Z_2$. If $\mathscr{M}$ is separable and k of its inseparable components have order twice their rank, then $\Gamma[BG(\mathscr{M})]$ is a supergroup of $\Gamma(\mathscr{M}_S)$ and a subgroup of $\Gamma(\mathscr{M}_S) \times \prod^{k} Z_2$.

4. BASIS GRAPHS WITH RESTRICTED CNs

We now analyze basis graphs in which only one or two types of CNs occur. Throughout we make use of the relationship between CNs and 2×2 submatrices of circuit matrices set forth in Theorem 2.2. For brevity, we call these submatrices *2-minors*.

The most interesting and simplest of our results is

THEOREM 4.1. *A matroid is binary if and only if its basis graph contains no induced octahedra.*

Proof. The only possible representation of a matroid by a combivalence system over F_2 is with its circuit matrices. These form a combivalence system iff (7) never occurs. (7) never occurs iff there are no octahedral CNs. In a basis graph any induced octahedral subgraph is necessarily a CN. $\blacksquare$

We will now assume that circuit matrices do not contain rows or columns of zeros. This assumption amounts to ignoring loops and coloops. It makes no essential difference in the theorems to follow, but does sometimes simplify their statements.

THEOREM 4.2. *All* CNs *of $BG(\mathscr{M})$ are squares if and only if $\mathscr{M}$ is representable by the combivalence system (over any field) of some $M = \sum M_i$ where each M_i is either an $m \times 1$ or a $1 \times n$ matrix of 1's.*

Proof. Let $\mathscr{P}$ be the pseudo-combivalence system of $\mathscr{M}$. We note that every CN of $BG(\mathscr{M})$ is a square iff every 2-minor of every $C(B) \in \mathscr{P}$ has two or fewer 1's. Also, if some $C(B_0) = \sum M_i$ with the M_i as above, then the pseudocombivalence rules have exactly the same effect on $\mathscr{P}$ as would combivalence rules over any field. Indeed, the only effect of pivoting is to change the border symbols, not the matrix entries. In particular, every 2-minor of every $C(B)$ would have two or fewer 1's. Now, if $\mathscr{M}$ is representable by a combivalence system in which some $M(B_0) = \sum M_i$, then $M(B_0) = C(B_0)$ and sufficiency is proved.

Conversely, suppose no 2-minor of $C(B_0)$ has more than two 1's. Rearrange the columns so that all the 1's in the first row are consecutive in the first p columns. If $p \geqslant 2$, there can be no 1's lower in those columns, else we get a 2-minor with $n \geqslant 3$ 1's. If $p = 1$, rearrange the other rows so that all the 1's in the first column are consecutive in the first q rows. If $q \geqslant 2$, there can be no 1's further right in rows 2 through q for the same reason as above. Thus we get (14).

$$\left[\begin{array}{c|c} 1 \cdots 1 & 0 \\ \hline 0 & M' \end{array}\right] \quad \text{or} \quad \left[\begin{array}{c|c} \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} & 0 \\ \hline 0 & M' \end{array}\right]. \tag{14}$$

Proceeding inductively, we get $M = \sum M_i$. ∎

COROLLARY 4.3. *If all* CNs *of* $G = BG(\mathscr{M})$ *are squares, then* G *is a product of complete graphs. If* G *contains no triangles, then it is an* n*-cube.*

Proof. Clearly each $m \times 1$ or $1 \times m$ matrix of 1's corresponds to a complete graph on $m + 1$ vertices. The first claim follows from the proof of Theorem 3.2. If there are no triangles, $m = 1$ in all cases and each complete graph is the interval K_2. By definition, a product of intervals is an n-cube. ∎

THEOREM 4.4. *All the* CN's *of* $BG(\mathscr{M})$ *are pyramids if and only if* $\mathscr{M}$ *can be represented by a combivalence system over* F_2 *containing a matrix entirely of* 1's.

Proof. Sufficiency: Suppose $\mathscr{M}(E, \mathscr{B})$ is representable over F_2 and some $M(B_0)$ is all 1's. Then each $c \in E - B_0$ can be interchanged with each $b \in B_0$. However, since all c's are represented by the same vector, at most one of them can appear in any given basis. In short, $\mathscr{B}$ consists of B_0 and all its neighbors, and $N(B_0)$ is "full." Therefore all CN's are pyramids; indeed, B_0 is the apex of every one.

Necessity: We first show that

$$\begin{array}{c|ccc|} & & & \\ b_1 & 1 & 1 & 1 \\ b_2 & 1 & 1 & 0 \\ \hline & c_1 & c_2 & c_3 \end{array} \tag{15}$$

cannot be the submatrix of any $C(B)$. Pivoting on (b_2, c_1) we get

$$\begin{array}{c|ccc|} & & & \\ b_1 & 1 & p & 1 \\ c_1 & 1 & 1 & 0 \\ \hline & b_2 & c_2 & c_3 \end{array}. \tag{16}$$

If $p = 1$, the first two columns of (15) and (16) correspond to an octahedron. If $p = 0$, the second two columns of (16) correspond to a square.

Now arrange $C(B)$ so that the row with the most 1's is on top and these 1's are consecutive in the first k columns. If there are no other rows (or not even one) we are done. If some other row has a 1 outside the first k columns, it must also have 1's in the first k columns as well; otherwise we get a submatrix

$$\begin{array}{|cc|} 1 & 0 \\ 0 & 1 \end{array}, \tag{17}$$

which gives a square CN. But then this row has more 1's than the first, which is also impossible. We conclude that there are no 1's beyond column k, and thus no columns beyond column k either.

Next suppose some row other than the first has at least two 1's. By the impossibility of (15) that row must consist entirely of 1's. Should some entry is some third row be 0, we get the transpose of (15), which is impossible by the same type of argument. Thus, if some row other than the first has two 1's, all entries of $C(B)$ are 1's.

If every row other than the first has just one 1, these must all occur in the same column; otherwise (17) occurs (up to order). If we pivot on the 1 in the first row and that column, we get a $C(B')$ which is all 1's.

Finally, since there are no octahedra, $\{C(B) | B \in \mathscr{B}\}$ must form a combivalence system over F_2. ▮

We note that we have also proved

COROLLARY 4.5. *If every* CN *of* $BG(\mathcal{M})$ *is a pyramid, then they all share the same apex.* ∎

THEOREM 4.6. *All the* CN's *of* $BG(\mathcal{M})$ *are octahedra if and only if* $\mathcal{M}$ *is full.*

Proof. Suppose each CN of $\mathcal{M}(E, \mathcal{B})$ is an octahedron. We claim that every $C(B)$ is a matrix of 1's. For suppose some $C(B)$ had a 0 in it. Since every row and column has at least one 1, we would get a submatrix

$$\boxed{\begin{matrix} 0 & 1 \\ 1 & p \end{matrix}}$$

and thus a CN with a missing vertex. Now pick $B \in \mathcal{B}$ and let S be any subset of E with $|B| = |S|$. Pick any $s \in S - B$ and $b \in B - S$. Then $B - b + s \in \mathcal{B}$ since $C(B)$ is all 1's. Continuing for $|B - S| - 1$ pivot steps more, we get $S \in \mathcal{B}$.

The converse is obvious. ∎

We now consider the cases in which one type of CN is excluded. The octahedral case was Theorem 4.1.

THEOREM 4.7. *A matroid has no pyramid* CNs *if and only if it is a sum of full matroids.*

Proof. No CN is a pyramid iff no 2-minor of any circuit matrix is (up to order)

$$\boxed{\begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}}\,. \tag{18}$$

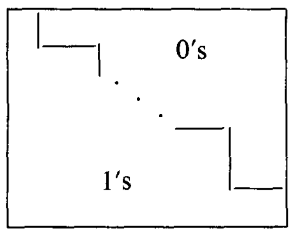
As for necessity, consider any $C(B)$ and let the 1's in the first row be consecutive in the first k columns. Any other row which has even one 1 in those columns must have 1's in exactly those columns, for otherwise (18) occurs. Thus, bringing such rows to the top, we get

$$\left[\begin{array}{c|c} \begin{matrix} 1 \cdots 1 \\ \cdot \quad \cdot \quad \cdot \\ 1 \cdots 1 \end{matrix} & 0 \\ \hline 0 & M' \end{array}\right]. \tag{19}$$

By induction, $C(B)$ breaks into blocks of 1's. The only way pivoting could cause a 1 to be replaced by a 0 would be (6), but this involves (18). Thus each block represents a full matroid.

As for sufficiency, each $C(B)$ is a sum of blocks of 1's, so (18) does not occur. ▌

We call a $0-1$ matrix *pseudo-triangular* if, up to order, it has the form (20). As long as we assume there are no zero rows or columns, the region of 1's must extend all the way to the top and the right.



(20)

THEOREM 4.8. *A matroid has no square* CNs *if and only if every one of its circuit matrices is pseudo-triangular.*

Proof. It suffices to prove that a matrix M is pseudo-triangular iff it does not contain (17). Necessity is clear. As for sufficiency, arrange the rows so that row i is above row i' if i has fewer 1's. Arrange the columns so that j is to the left of j' if it has more 1's. If row i has a 1 in column p, it must have 1's in all the columns to the left of p; if it had a 0 in column $n < p$, by the column arrangement we would obtain (17) after all (up to order). Thus by the row arrangement, M is pseudo-triangular. ▌

This characterization of matroids without square CNs is quite artificial. We now present an interesting characterization for a certain subclass, but unfortunately the subclass is proper.

Let $E_1, E_2, ..., E_k$ be subsets of a finite set E. A subset $\{e_1, ..., e_j\}$ of E is said to be a *system of distinct representatives* (SDR) if there is an injection $e_i \to E_{n(i)}$ where $e_i \in E_{n(i)}$. Let $\mathscr{B}$ be the collection of maximal SDRs. As shown by Edmonds and Fulkerson [2], $(E, \mathscr{B})$ is a matroid, a *transversal matroid.* Note that this definition allows for loops and coloops, as does the material to follow.

A transversal matroid is *(properly) nested* if the E_i are (properly) nested by inclusion. There is a simple mapping, due to Welsh [14], between $0-1$ sequences $a_1a_2 \cdots a_n$ and nested transversal matroids. Namely, let

$E = \bar{n} = \{1, 2,..., n\}$ and let j be an E_i iff $a_j = 1$. Clearly any properly nested collection $E_1 \subset E_2 \cdots \subset E_k$ can be obtained in this way by first numbering the elements of E consecutively, starting with those of E_1, then $E_2 - E_1$, and ending with $E - E_k$. Below we assume such a numbering has been made.

It is easy to show that every nested transversal matroid is isomorphic to a properly nested one; indeed, every transversal matroid is isomorphic to one with distinct E_i. Thus Welsh's correspondence is essentially a surjection. As he showed, it is also an injection. An alternate proof follows from the comment after (23) below.

THEOREM 4.9. *Nested transversal matroids have no square* CNs.

Proof. If there were a square then some $C(B)$ would have a submatrix

$$\begin{array}{c|cc|} k & 1 & 0 \\ l & 0 & 1 \\ \hline & i & j \end{array}, \tag{21}$$

where we may assume $i < j$. Because of the nesting and ordering, a lower number not being used as a representative can always replace a higher number which is. Since $B - l + j$ is a maximal SDR, $B - l + i$ must thus be one too, contradicting (21). ∎

Suppose $\mathscr{M}(E, \mathscr{B})$ arises from the proper nest $E_1 \subset E_2 \cdots \subset E_k$. If e_j is the smallest element of $E_j - E_{j-1}$, we call the maximal SDR $B_0 = \{e_1 ,..., e_k\}$ *standard.* When the elements of B_0 and $E - B_0$ are arranged in order, $C(B_0)$ is pseudo-triangular. For instance, if $|E| = 6$, $k = 3$ and

$$E_1 = \bar{3}, \qquad E_2 = \bar{5}, \qquad E_3 = \bar{6}, \tag{22}$$

then $C(B_0)$ is

$$\begin{array}{c|ccc|} 1 & 1 & 1 & 0 \\ 4 & 1 & 1 & 1 \\ 6 & 1 & 1 & 1 \\ \hline & 2 & 3 & 5 \end{array}. \tag{23}$$

One can show that any other $C(B)$ of $\mathscr{M}$, when suitably arranged, has 1's in at least all the locations where they occur in $C(B_0)$.

Now let $\mathcal{M}'$ be obtained from a full matroid of order 6 and rank 3 by deleting any two bases distance 3 apart. By Theorem 1.3, $\mathcal{M}'$ is a matroid. Clearly each $C(B)$ of $\mathcal{M}'$ contains at most one 0. If $\mathcal{M}'$ were a nested transversal matroid, its standard circuit matrix would thus be (23) and its E_i would be (22). But there is only *one* 3-subset in that system which is not a maximal SDR, $\{4, 5, 6\}$. Thus $\mathcal{M}'$ is not a nested transversal matroid.

5. Polars

Whereas the top level $\mathscr{B}_0$ of a leveled basis graph has only a single vertex, we have seen that the bottom may have many. Also, two adjacent vertices in one level must have a common neighbor in the level up (Lemma I.2.8) but not necessarily in the next level down. Nonetheless, there is much symmetry between up and down. There is also much structure in the top and bottom levels, which we call *polars*, and these tie in nicely with other matroid concepts.

We begin by generalizing levelings in a way which makes the symmetry clear. Given $\mathcal{M}(E, \mathscr{B})$ and $E' \subset E$, let

$$M(E') = \max_{\mathscr{B}} | B \cap E' |, \qquad m(E') = \min_{\mathscr{B}} | B \cap E' |.$$

Then a *matroid leveling* of $\mathcal{M}(E, \mathscr{B})$ with index E' is a partition of $\mathscr{B}$ into $\mathscr{B}_k$, $k = 0, 1, \ldots$, where

$$\mathscr{B}_k = \{B \mid M(E') - | B \cap E' | = k\}. \tag{24}$$

If $E' = B \in \mathscr{B}$, this is precisely the leveling from B we have used previously. In general, we call the top level $\mathscr{B}_0(E')$ and the bottom $\mathscr{B}^0(E')$. Clearly the matroid leveling from $E - E'$ is just the leveling from E' turned upside down. In particular,

$$\mathscr{B}_0(E - E') = \mathscr{B}^0(E'). \tag{25}$$

This leveling generalization is closely related to one of the standard definitions of minors. The *reduction* of $\mathcal{M}(E, \mathscr{B})$ to E', written $\mathcal{M} \cdot E'$, is the matroid $(E', \mathscr{B}')$ where

$$\mathscr{B}' = \{B \cap E' \mid \ B \in \mathscr{B}, \ \ | B \cap E' | = M(E')\}.$$

The *contraction* $\mathcal{M} \times E''$ is $(E'', \mathscr{B}'')$ where

$$\mathscr{B}'' = \{B \cap E'' \mid \ B \in \mathscr{B}, \ \ | B \cap E'' | = m(E'')\}.$$

A *minor* of $\mathscr{M}$ is any matroid that can be obtained from $\mathscr{M}$ by a series of reductions and contractions. It is well known (and implied by the next proof) that $\mathscr{M} \cdot E'$ and $\mathscr{M} \times E'$ are in fact matroids. Also, any minor can be obtained by at most one reduction followed by at most one contraction, or vice versa.

From the definitions it is clear that $\mathscr{B}_0(E')$ is a subset of the bases of $\mathscr{M} \cdot E' + \mathscr{M} \times (E - E')$. In fact, we have

THEOREM 5.1. *$(E, \mathscr{B}_0(E'))$ is a matroid $\mathscr{M}(E')$, and*

$$\mathscr{M}(E') = \mathscr{M} \cdot E' + \mathscr{M} \times (E - E').$$

This is essentially proposition 3.53 in Tutte's lectures [13], but we give another proof. For any $B \in \mathscr{B}_0(E')$ let

$$B' = B \cap E', \qquad B'' = B \cap (E - E'),$$
$$C' = (E - B) \cap E', \qquad C'' = (E - B) \cap (E - E').$$

Then the circuit matrix at B must be of the form

$$\begin{array}{c} B' \\ B'' \end{array} \left[\begin{array}{c|c} M_1 & M_3 \\ \hline 0 & M_2 \end{array}\right]_{\substack{\\ C' \quad C''}}; \tag{26}$$

for, if some entry in the lower left were 1, then $| B' |$ would not equal $M(E')$. By the pseudo-combivalence rules, no matter what M_3 is, no pivot in M_1 affects M_2, and vice versa. Also, pivoting in M_3 takes us out of $\mathscr{B}_0(E')$, and this is never needed to reach another $\hat{B} \in \mathscr{B}_0(E')$. For instance, we can pivot in all the elements of $(\hat{B} - B) \cap E'$ first and then pivot out $(B - \hat{B}) \cap E''$; this necessarily avoids pivoting in the upper right and gets us from B to $\hat{B}$. Thus, as far as $\mathscr{B}_0(E')$ is concerned, we may set $M_3 = 0$. ∎

By symmetry, $\mathscr{B}^0(E')$ forms a matroid

$$\mathscr{M}(E')^0 = \mathscr{M} \times E' + \mathscr{M} \cdot (E - E').$$

Considering any natural image of $\mathscr{M} \cdot E'$ in $\mathscr{M}(E')$, and any of $\mathscr{M} \times E'$ in $\mathscr{M}(E')^0$, we see that reductions and contractions are, except for loops and coloops, submatroids of $\mathscr{M}$. Since submatroids of submatroids are submatroids, every minor is a submatroid in this sense. However, the converse is false; unless its factorization is trivial, $(E, \mathscr{B}_0(E'))$ is not a minor. In the next section, it is shown that minors may be distinguished from submatroids in general by the nature of their subgraphs in $BG(\mathscr{M})$.

Situations in which the factorization of $\mathscr{M}(E')$ is trivial are not without interest. A *circuit* of $\mathscr{M}(E, \mathscr{B})$ is a subset of E contained in no basis and minimal with respect to this property. A *cocircuit* (cut-set) is a minimal subset among those that intersect every basis.

COROLLARY 5.2. *Suppose $E - E'$ contains neither a circuit not a cocircuit. Then ignoring loops and coloops, $\mathscr{M} \cdot E'$ and $\mathscr{M} \times E'$ are polars.*

Proof. From the definitions, $\mathscr{M} \times (E - E')$ and $\mathscr{M} \cdot (E - E')$ are trivial. ▌

As an example, suppose edge e of graph G is neither a loop nor a bridge. Let $\mathscr{M}$ be the forest matroid of G (see next section). Then the matroids of the graphs obtained by deleting and contracting e are polars of $\mathscr{M}$. As noted by many people, their bases partition those of $\mathscr{M}$.

Clearly there are three more corollaries analogous to the one above; one merely trivializes a different pair of factors from $\mathscr{M}(E')$ and $\mathscr{M}(E')^0$. We go on to duality.

THEOREM 5.3. $[\mathscr{M}(E')]^* = [\mathscr{M}^*(E')]^0 = \mathscr{M}^*(E - E')$.

Proof. The second equality is an instance of (25). As for the first, B is a basis of $\mathscr{M}(E')$ iff $| B \cap E' |$ is maximal for $\mathscr{M}$. But this is clearly equivalent to $|(E - B) \cap E'|$ being minimal for $\mathscr{M}^*$. ▌

As for the non-polar regions, our only result is

THEOREM 5.4. *Let $BG(E, \mathscr{B})$ be leveled with index E'. Then the induced subgraph on $\mathscr{B}_k \cup \mathscr{B}_{k+1}$ is connected.*

Proof. Let $B = A \cup D$ and $B' = A' \cup D'$ be distinct vertices in $\mathscr{B}_k \cup \mathscr{B}_{k+1}$, where $A, A' \subset E'$ and $D, D' \subset E - E'$. By up-down symmetry we may suppose $B \in \mathscr{B}_k$. If $D' - D = \varnothing$, we must have $D' = D$; otherwise B' is in some $\mathscr{B}_j$, $j < k$. Thus applying the exchange axiom to B, B' gives $B'' = B - a + a' \in \mathscr{B}_k$. If $D' - D \neq \varnothing$, pick any $d' \in D' - D$; for some $b \in B - B'$, we get $B'' = B - b + d' \in \mathscr{B}$. If $b \in A$, then $B'' \in \mathscr{B}_{k+1}$. If $b \in D$, $B'' \in \mathscr{B}_k$. In all three cases $B'' \in \mathscr{B}_k \cup \mathscr{B}_{k+1}$ and $| B'' - B' | = | B - B' | - 1$. The result follows by induction. ▌

In general, $\mathscr{B}_k$ is not connected and $(E, \mathscr{B}_k \cup \mathscr{B}_{k+1})$ is not a submatroid.

Finally, we compare matroid levelings to another generalization of the original levelings, one which makes sense for any graph. Given $v \in \mathscr{V}(G)$ and $\mathscr{V}' \subset \mathscr{V}$, let

$$\delta(v, \mathscr{V}') = \min_{\mathscr{V}'} \delta(v, v').$$

Then the *leveling* from $\mathscr{V}'$ is a partition of $\mathscr{V}$ into sets

$$\mathscr{V}_k = \{v \mid \delta(v, \mathscr{V}') = k\}.$$

THEOREM 5.5. *Every matroid leveling is a leveling.*

Proof. We must show that in the matroid leveling (24) of $\mathscr{M}(E, \mathscr{B})$ from E',

$$\mathscr{B}_k = \{B \mid \delta(B, \mathscr{B}_0) = k\}.$$

If $B_0 \in \mathscr{B}_0$, $B_k \in \mathscr{B}_k$, then $| B_0 \cap E' | - | B_k \cap E' | = k$. Let

$$E_1 = (B_0 - B_k) \cap E', \qquad E_2 = (B_k - B_0) \cap E'.$$

Then $| E_1 | = m \geqslant k$ and $| E_2 | = m - k$. In particular, $\delta(B_k, \mathscr{B}_0) \geqslant k$. By the exchange axiom we may forge a path from B_k to B_0 so that for each edge one element of $B_k - B_0$ is pivoted out and one of $B_0 - B_k$ is pivoted in. Let B^j be the $j + 1$ basis in this path, e.g., $B^0 = B_k$. Also by the exchange axiom, we may assume that the elements of E_1 are pivoted in first, that is, one for each edge of $B^0 \cdots B^m$. Along the same subpath any number of elements of E_2 may be pivoted out, but by applying the exchange axiom to B^m and B^0 we may assume they are pivoted out last. Thus $B^k \in \mathscr{B}_0$, and $\delta(B_k, \mathscr{B}_0) \leqslant k$. ∎

By symmetry we get the following corollary, which for matrix matroids amounts to a well-known result about pivoting.

THEOREM 5.6. *Given $\mathscr{M}(E, \mathscr{B})$, suppose one starts at some B_0 and moves through any sequence of bases which involves at each step exchanging an element of B_0 for one of $E - B_0$. When the sequence can no longer be continued, the basis one has is in $\mathscr{B}^0(B_0)$.*

Proof. One could not possibly get stuck at some B' outside $\mathscr{B}^0(B_0)$, for, by the proof above, there is a direct path from B' to

$$\mathscr{B}_0(E - B_0) = \mathscr{B}^0(B_0)$$

on which one could continue. ∎

In other words, if one wants to throw out as many elements of B_0 as possible, one may do so by charging directly ahead and without any advanced planning. This fact is useful in linear algebra.

6. Regular and Graphic Basis Graphs

Recall that $\mathscr{M}(E, \mathscr{B})$ is *graphic* if E is the set of edges of some graph G and $\mathscr{B}$ is its set of spanning forests. We write $\mathscr{M} = \mathscr{M}(G)$. $\mathscr{M}$ is *cographic* if $\mathscr{M}^*$ is graphic. $\mathscr{M}(G)$ is planar if G is. By a celebrated theorem of Whitney (translated into matroid terminology), $\mathscr{M}$ is planar iff it is graphic and cographic.

There are many equivalent definitions of *regular* matroids. From our point of view the most interesting is that $\mathscr{M}$ have a combivalence representation over the rationals in which every entry of every $M(B)$ is either 0, 1, or -1. See Rockafellar [9, Section 6] for a discussion of the various definitions.

Tutte [13] has characterized the classes of matroids above in terms of forbidden minors. To use his results, we must make precise the relation between minors of $\mathscr{M}$ and subgraphs of $BG(\mathscr{M})$. The induced subgraph $\langle \mathscr{V}' \rangle$ of $G(\mathscr{V}, \mathscr{E})$ is an SPC (*shortest path complete*) if $\mathscr{V}'$ satisfies the following condition: whenever $v \in \mathscr{V}$ is on some shortest path of G between $v', v'' \in \mathscr{V}'$, then $v \in \mathscr{V}'$. Clearly every CN is an SPC. Moreover, if $\langle \mathscr{V}' \rangle$ is an SPC, then $\mathrm{CN}(v', v'')$ in $\langle \mathscr{V}' \rangle$ is the same as $\mathrm{CN}(v', v'')$ in G. By Theorem 1.1 there is a submatroid $\mathscr{M}'$ such that $\langle \mathscr{V}' \rangle = BG(\mathscr{M}')$. In fact,

THEOREM 6.1. *A subgraph G' of $BG(\mathscr{M})$ is an* SPC *if and only if it is the labeled basis graph of a submatroid which is, except for loops and coloops, a minor.*

Proof. First we show that $\mathscr{M}'(E', \mathscr{B}')$ is a minor of $\mathscr{M}(E, \mathscr{B})$ iff there exists a partition $E = E' \cup L \cup C$ such that $\mathscr{B}'$ equals

$$\{B \cap E' \mid B \in \mathscr{B}, \quad C \subset B, \quad B \cap L = \varnothing\}. \tag{27}$$

Sufficiency is easy: $\mathscr{M}' = [\mathscr{M} \cdot (E - L)] \times E'$. As for necessity, suppose $\mathscr{M}' = \mathscr{M} \cdot E'$. Let C be any basis in $\mathscr{M} \times (E - E')$ and let $L = E - E' - C$. By definition, $\mathscr{B}'$ is just the projection of $\mathscr{B}_0(E')$ under the mapping $S \to S \cap E'$. Since $\mathscr{M}(E') = \mathscr{M} \cdot E' + \mathscr{M} \times (E - E')$, $\mathscr{B}'$ is also the projection of any cross-section of $\mathscr{B}_0(E')$ arising from a fixed basis of $\mathscr{M} \times (E - E')$. (27) is just such a projection.

If $\mathscr{M}' = \mathscr{M} \times E'$, analogous reasoning applies. In all remaining cases

$\mathscr{M}' = (\mathscr{M} \cdot E'') \times E'$ and we apply the special case twice. $E'' - E'$ partitions into L', C' such that the bases of $\mathscr{M}'$ are the restrictions to E' of the bases in $\mathscr{M} \cdot E''$ which include C' and exclude L'. $E - E''$ partitions into L'', C'' with similar properties in regard to $\mathscr{M} \cdot E''$ and $\mathscr{M}$. Thus the bases of $\mathscr{M}'$ are (27) with $L = L' \cup L''$ and $C = C' \cup C''$.

Now we show that an induced subgraph G' of $BG(\mathscr{M})$ is an SPC iff there exist L, C such that $\mathscr{V}' = \mathscr{V}(G')$ is just

$$\mathscr{B}' = \{B \in \mathscr{B} \mid C \subset B, B \cap L = \varnothing\}. \tag{28}$$

From this and the first claim the theorem follows.

Suppose $\mathscr{V}' = \mathscr{B}'$. Let $B_0 \cdots B_n$ be a shortest path between $B_0, B_n \in \mathscr{V}'$. The only elements pivoted out (in) along such a path are those of $B_0 - B_n$ $(B_n - B_0)$. Thus each vertex on the path is in $\mathscr{B}' = \mathscr{V}'$.

Conversely, suppose G' is an SPC. Let C be the elements which occur in all $B \in \mathscr{V}'$ and L those which occur in none. Clearly $\mathscr{V}' \subset \mathscr{B}'$. Suppose B' were in $\mathscr{B}'$ but not in $\mathscr{V}'$. Since any shortest path from B' to any $B_0 \in \mathscr{V}'$ is entirely in $\mathscr{B}'$, we may assume B' is adjacent to B_0, that is, $B' = B_0 - e_1 + e_2$. Because $e_i \notin L \cup C$, there exist $B_2, B_3 \in \mathscr{V}'$ such that $e_1 \notin B_2$, $e_2 \in B_3$. Since $(E, \mathscr{V}')$ is a matroid, by the exchange axiom we may assume $B_2 = B_0 - e_1 + e'$ and $B_3 = B_0 - e'' + e_2$. But then $CN(B_2, B_3)$ includes B' so $B' \in \mathscr{V}'$ after all. ∎

Incidentally, we have shown that $\langle \mathscr{V}' \rangle$ is an SPC iff it is connected and every CN(v', v'') in it is identical to CN(v, v'') in $BG(\mathscr{M})$.

By one of Tutte's theorems, a matroid is regular iff it is binary and no minor corresponds to the F_2 combivalence system of

$$\boxed{\begin{matrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{matrix}}$$

or its transpose. The matroids of these two systems are duals. Thus they have the same unlabeled basis graph, call it $\hat{G}$. They are also inseparable, so, by Theorem 3.4, any matroid with basis graph $\hat{G}$ differs from one or the other merely by loops and coloops. We have

THEOREM 6.2. *$\mathscr{M}$ is regular if and only if no* SPC *of $BG(\mathscr{M})$ is an octahedron or $\hat{G}$.* ∎

Unfortunately, $\hat{G}$ has 29 vertices, so we do not find this a very useful characterization.

As for basis graph characterizations of graphic matroids, it is easy to see that there are none. Any condition on basis graphs which would accept $\mathscr{M}(G)$ would also accept $\mathscr{M}(G)^*$, which need not be graphic. However, we can attack the related problems of characterizing matroids which are graphic and (or) cographic. Tutte has shown that $\mathscr{M}$ is graphic iff it is regular and no minor is $\mathscr{M}(K_{3,3})^*$ or $\mathscr{M}(K_5)^*$. Likewise, $\mathscr{M}$ is cographic iff it is regular and no minor is $\mathscr{M}(K_{3,3})$ or $\mathscr{M}(K_5)$. Let $G_{3,3}$ and G_5 represent the two basis graphs of these four matroids. Then

THEOREM 6.3. *$\mathscr{M}$ is planar if and only if no* SPC *of $BG(\mathscr{M})$ is an octahedron, $\hat{G}$, $G_{3,3}$, or G_5.* ▌

To characterize the class of matroids which are either graphic or cographic, we must modify the above to allow occurrences of $G_{3,3}$ and G_5 as long as they are all labeled dually, i.e., with $\mathscr{M}(K_{3,3})^*$ and $\mathscr{M}(K_5)^*$, or all not. This uniformity of labelings can be expressed graph-theoretically in terms of chains of cliques in $BG(\mathscr{M})$; see [1, Theorem 2]. However, $G_{3,3}$ has 63 vertices and G_5 has 125, so we see no point in working out this characterization precisely.

Two final remarks: First, in light of the results above, we doubt that basis graphs provide a fruitful context in which to attack the representability problem.

Second, we have used Theorem 6.1 in one direction only. Our basis graph characterization of binary matroids provides an opportunity to go the other way. We get that $\mathscr{M}$ is binary iff the full matroid of order 4 and rank 2 is not a minor.

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